Bi-Partite Graphs with Theorems

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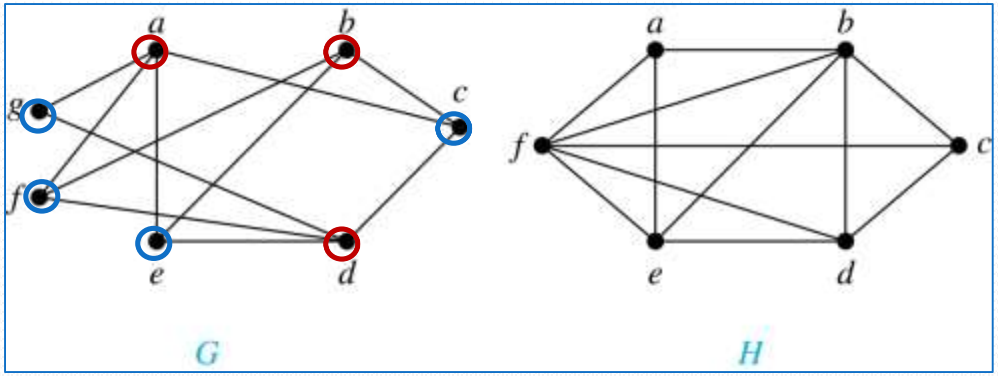
Bipartite Graphs

Definition: A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

It is not hard to show that an equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

G is bipartite

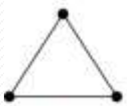


H is not bipartite since if we color *a* red, then the adjacent vertices *f* and *b* must both be blue.

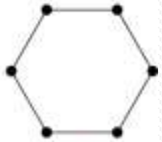
Bipartite Graphs (*continued*)

Example: Show that C_6 is bipartite.

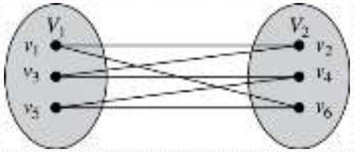
Solution: We can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ so that every edge of C_6 connects a vertex in V_1 and V_2 .



C_3



C_6



Example: Show that C_3 is not bipartite.

Solution: If we divide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Bipartite Graphs (*Theorems*)

- **Theorem**: *A bipartite graph contains no odd cycles.*

  Proof:

  If G is bipartite, let the vertex partitions be X and Y . Suppose that G did contain an odd cycle – then C = $v_0 e_1 \ldots e_{2k+1} v_0$.
  Without loss of generality, let v_0 be a vertex in X. Then v_1 must be a vertex in Y , and it is connected to v_0 by e_1.
  Similarly, e_{2n+1} is preceded by a vertex in X and proceeded by a vertex in Y for all n N . But e_{2k+1} is proceeded by v_0, which is a vertex in X and therefore cannot also be a vertex in Y .

  In fact, any graph that contains no odd cycles is necessarily bipartite, as well. This we will not prove, but this theorem gives us a nice way of checking to see if a given graph G is bipartite – we look at all of the cycles, and if we find an odd cycle we know it is not a bipartite graph.

Bipartite Graphs (*Theorems*)

- **Theorem**: (Sub-graph of a Bipartite Graph) *Every subgraph H of a bipartite graph G is itself bipartite.*

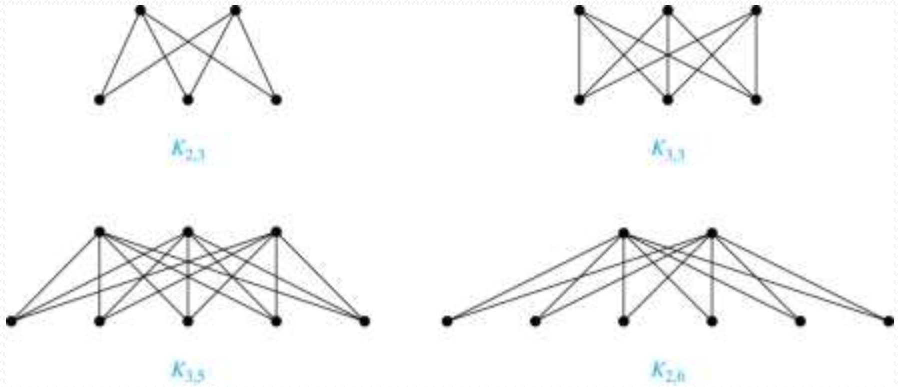
  Proof:

- If G is bipartite, let the partitions of the vertices be X and Y. Then let $X^{J} = X$ intersection H and $Y^{J} = Y$ intersection H. Suppose that this was not a valid bipartition of H – then we have that there exists v and u in X^{J} (without loss of generality) such that v and u are adjacent. But then by the definition of a subgraph, they are also adjacent
- in G. But then X and Y is not a valid bipartition of G. Therefore, H is a bipartite graph.

Complete Bipartite Graphs

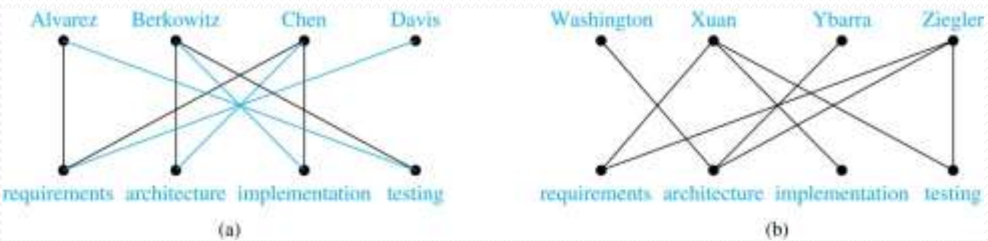
Definition: A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Example: We display four complete bipartite graphs here.



Bipartite Graph applications

- Bipartite graphs are used to model applications that involve matching the elements of one set to elements in another, for example:
- *Job assignments* - vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done.



- *Marriage* - vertices represent the men and the women and edges link a a man and a woman if they are an acceptable spouse. We may wish to find the largest number of possible marriages.